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Steady subcritical thermohaline convection

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Steady convective motions in a Boussinesq fluid with an unstable thermal and stable salinity stratification are investigated in the case that the ratio of diffusivities $\tau \equiv \kappa_S/\kappa_T \ll 1$. Using perturbation theory, it is shown that, for any value of the salt Rayleigh number R_S, finite-amplitude convection can occur at values of the Rayleigh number R_T much less than that necessary for infinitesimal oscillations, provided only that τ is sufficiently small. A simple qualitative argument is used to show how $R_{\min}$, the minimum value of R_T for steady convection, varies with R_S, and it is shown that the analytical results of the present paper form a natural complement to the numerical ones of Huppert & Moore (1976). Results are presented both for stress-free and for rigid boundaries, and applicability of the method to other related problems is suggested.

1. Introduction

The phenomenon of double-diffusive convection in a fluid layer, where two scalar fields (such as heat and salinity concentration) affect the density distribution in a fluid, has become increasingly important (and widely studied) in recent years. The classical Bénard problem, with no salt, was first treated by Rayleigh (1916) and laminar convection is now well understood. The variety of behaviour in the double-diffusive case is much greater than that for the Bénard problem. Linearized stability theory (Baines & Gill 1969) shows, in particular, that the first occurrence of instability can take the form of oscillations rather than direct convection if the component with the smaller diffusivity is stably stratified, provided that R_S (a dimensionless measure of this stratification) is sufficiently large. Finite-amplitude convection was discussed by Veronis (1965, 1968). He showed that when oscillatory convection was possible there was always an unstable branch of steady solutions bifurcating from the static state at larger values of R_T (a measure of the destabilizing gradient of the larger diffusivity component). He then used a truncated Fourier series representation of the solutions to obtain a guide to the finite-amplitude behaviour of the steady solution branch. The results suggested that steady motion at finite amplitude could occur at values of R_T much less than that predicted by linearized theory. Indeed, within the context of the modal expansion the minimum value $R_{\min}$ of R_T for which steady solutions are possible is always less than R_0, the value of R_T for which the bifurcation to oscillatory convection occurs. (This result has been clarified by Da Costa, Knobloch & Weiss (1981), who conduct an extensive numerical investigation of the truncated model equations.)

More recently Huppert & Moore (1976; hereinafter referred to as HM) have conducted a comprehensive numerical study of the full equations of motion in a two-dimensional geometry. Among many results, they find that the sign of $R_0 - R_{\min}$ is not always positive, and that in fact $R_{\min}$ is less than R_0 only for rather special values of the parameters. In particular, if $\tau = \kappa_S/\kappa_T$ (the ratio of the diffusivities of heat and salt) is very small (it will be recalled that only for $\tau < 1$ is overstability possible) then $R_{\min} < R_0$ for moderate values of R_S. The precise nature of the criterion was not completely clarified, however, and in view of its importance we felt that a closer look was necessary, especially for very small τ (which HM's numerical scheme was unable to treat accurately).

In this paper, then, we investigate the case of small τ analytically, making use of boundary-layer analysis to describe the salt field, following Roberts (1979). The analysis proceeds similarly to that of Busse (1975) and Proctor & Galloway (1979), who investigated the allied problem of convection in an imposed magnetic field. It emerges that the key parameter in the analysis is $R_S\tau^{\frac{1}{2}}$ (for rigid boundaries, the relevant parameter is $R_S\tau^{\frac{2}{3}}$) which must be small for the validity of the results; R_S can still be $O(1)$, however, so that overstable oscillations can exist. (When $R_S\tau^{\frac{1}{2}}$ is not small a full solution by this method is not possible, but we attempt to show via a qualitative model that $R_{\min}$ and R_S are approximately linearly related when $R_S\tau^{\frac{1}{2}} \gg 1$. Such a relation was in fact found by HM in their numerical study.) This procedure enables us to treat both stress-free boundaries (discussed by HM) and rigid boundaries, on which no work appears to have been done. An interesting outcome of the analysis (when $R_S\tau^{\frac{1}{2}} \ll 1$) is that, whatever R_S may be, steady finite-amplitude convection can occur at values of R_T arbitrarily close to that necessary for normal Bénard convection ($R_S = 0$) as $\kappa_S \to 0$ with all the other parameters kept fixed. A similar result was found by Gough (unpublished) using a modal representation of convection.

The plan of the paper is as follows. In §2 the problem is formulated, and the approximations that are employed are made explicit. The asymptotic analysis for small τ is carried out in §3, and solution for the case of free boundaries undertaken in §4. In §5 we treat the case of fixed boundaries, and discuss the results in §6. We conclude in §7 with a description of the qualitative model for higher values of R_S.

2. Formulation and derivation of the expansion scheme

The dimensionless equations describing steady two-dimensional convection in a Boussinesq double-diffusive fluid may be written (see, for example, HM)

$$\sigma^{-1}(\mathbf{U}.\nabla\omega) = -R_T\frac{\partial\theta}{\partial x} + R_S\frac{\partial S}{\partial x} - \nabla^4\psi, \tag{2.1}$$

$$\mathbf{U}.\nabla\theta = \nabla^2\theta, \tag{2.2}$$

$$\mathbf{U}.\nabla S = \tau\nabla^2 S, \tag{2.3}$$

where $\mathbf{U}(x,z) = (-\partial\psi/\partial z, 0, \partial\psi/\partial x)$ in Cartesian co-ordinates (x,y,z) and $\partial/\partial y = 0$.

The layer depth is scaled with d, the velocity $\mathbf{U}$ with κ_T/d, where κ_T is the diffusivity of heat. The perturbation temperature and salinity are related to the corresponding dimensional (starred) quantities by

$$\theta^* = T_0 + \theta\Delta T, \quad S^* = S_0 + S\Delta S. \tag{2.4}$$

Here S_0 and T_0 are basic salinity and temperature values supposed to hold at $z = \frac{1}{2}$, the mid-point of the layer, with ΔT, ΔS the difference in the temperature and salinity concentrations across the layer, so that at $z = 0, 1, \theta = S = \pm\frac{1}{2}$ respectively. The y component of vorticity $\omega = -\nabla^2\psi$.

The dimensionless parameters are:

the Prandtl number $\sigma = \nu/\kappa_T$; (2.5)

the Schmidt number $\tau = \kappa_S/\kappa_T$; (2.6)

the Rayleigh number $R_T = \dfrac{g\alpha\Delta T d^3}{\kappa_T \nu}$; (2.7)

and the salt Rayleigh number $R_S = \dfrac{g\beta\Delta S d^3}{\kappa_T \nu}$. (2.8)

It will be convenient to define a further parameter

$$P_S = R_S/\tau. \tag{2.9}$$

Note that P_S is independent of κ_T. The coefficients α and β are those that appear in the assumed linear density relation $\rho = \rho_0(1-\alpha\theta\Delta T+\beta S\Delta S)$. We suppose that the top and bottom boundaries of the layer are either stress-free (as assumed by HM) or rigid, as is more likely in experiments. Thus we have, at $z = 0, 1$,

$$\psi = \psi_{zz} = 0 \quad \text{(stress-free)}, \tag{2.10}$$

$$\psi = \psi_z = 0 \quad \text{(rigid)}, \tag{2.11}$$

where the subscripts denote partial derivatives.

We will suppose that the motion is periodic in the x direction with period $2\pi/k$. Thus at $x = 0, k$ we have, by symmetry,

$$\psi = \psi_{xx} = 0, \quad \theta_x = S_x = 0, \tag{2.12}$$

and these are the conditions used by HM. In order to make analytical progress, we shall suppose that $R_T = O(1)$ and that the Péclet number (proportional to $|\mathbf{U}|$ in the present scaling) is small. This means that the isotherms are not greatly distorted from their original horizontal form in the absence of motion. The Prandtl number σ is supposed $O(1)$, and τ supposed small (for example, $\tau \simeq \frac{1}{80}$ for water). The assumptions imply restrictions on the size of R_S which will energe in the course of the analysis.

3. The asymptotic problem when $\tau \ll 1$

It is well known that (2.1)–(2.3) allow a basic hydrostatic state, namely $\mathbf{U} = 0$, $S = \theta = \frac{1}{2}-z$. If R_T is large enough then convective solutions ($\mathbf{U} \neq 0$) become possible. We will suppose that R_T is such that the motions are 'of small amplitude' in the sense that the Péclet number is small. Then we can write

$$\mathbf{U} = \epsilon\mathbf{U}_1+\epsilon^2\mathbf{U}_2+\dots, \tag{3.1}$$

$$\theta = \tfrac{1}{2}-z+\epsilon\theta_1+\epsilon^2\theta_2+\dots, \tag{3.2}$$

$$R_T = R_0+\epsilon R_1+\epsilon^2 R_2+\dots, \tag{3.3}$$

with ψ_1, ψ_2...defined in the obvious manner, and where the small parameter ϵ can be identified with the Péclet number and $\mathbf{U}_1$, θ_1 are of order unity. The purpose of the analysis is to find ϵ in terms of R_T. We fix the size of ϵ by requiring of the leading-order functions $\mathbf{U}_1$, θ_1 that

$$\frac{1}{k}\int_0^k\int_0^1 \theta_1 \frac{\partial \psi_1}{\partial x}\,dx\,dz = 1. \tag{3.4}$$

It should be noted that S is not expanded in powers of ϵ, since the assumption of small τ means that S may differ significantly from the static configuration, even when ϵ is small. In fact, we shall suppose that ϵ is large enough and τ small enough that $\tau \ll \epsilon$. Thus the situation we envisage is one in which the isotherms are scarcely disturbed (small Péclet number), but the isopycnals (constant S) are greatly affected by the flow. Clearly if τ is small enough there is a range of ϵ such that $1 \gg \epsilon \gg \tau$, and it is this range on which we wish to focus.

We then define a parameter $\eta = \tau\epsilon^{-1}$, which by our supposition is small, and substitute (3.1)–(3.3) into the governing equations. This procedure yields at leading order

$$0 = R_0 \frac{\partial \theta_1}{\partial x} - P_S \eta \frac{\partial S}{\partial x} + \nabla^4 \psi_1, \tag{3.5}$$

$$0 = \frac{\partial \psi_1}{\partial x} + \nabla^2 \theta_1, \tag{3.6}$$

$$\mathbf{U}_1 . \nabla S = \eta \nabla^2 S. \tag{3.7}$$

We note that the effects of finite Péclet number do not appear in this system, since if the term in P_S is ignored equations (3.5)–(3.6) are linear, and constitute the standard linearized stability problem for Rayleigh–Bénard convection with eigenvalue R_0. The effect of finite ϵ if $P_S = 0$ can be shown to result in an expansion of the form

$$R_T = R_0 + \epsilon^2 R_2 + \ldots, \tag{3.8}$$

where R_2 is a positive constant that depends on σ and the boundary conditions (see, for example, Malkus & Veronis 1958). Thus in the absence of salt R_T increases monotonically with ϵ for small ϵ. In order to find a parameter regime in which R_T has a minimum as a function of ϵ, we must consider additionally the effect of non-zero P_S on R_0. If this effect is *small* (i.e. if P_S is not too large) then the effects of salt and finite Péclet number on R_0 can be added together, correct to leading order. All these ideas are in the spirit of Busse (1975), who considered the similar magnetoconvection problem.

If the effects of salt are included in (3.5)–(3.7) then R_0 is no longer an eigenvalue, but depends on P_S and on η. If $R^{(0)}$ is the eigenvalue for the non-salt problem ($P_S = 0$), we suppose that the effect of the salt is small, and write

$$\left.\begin{aligned} R_0 &= R^{(0)} + \delta R^{(1)} + \ldots, \\ S &= S^{(0)} + \delta S^{(1)} + \ldots, \\ \theta_1 &= \theta^{(0)} + \delta\theta^{(1)} + \ldots, \end{aligned}\right\} \tag{3.9}$$

and similarly for $\mathbf{U}_1$, ψ_1, where $R^{(1)}$ is independent of P_S and η and δ is a small parameter that is a function of P_S and η alone (the precise function depends, as we shall see, on the boundary conditions). Then we expand all variables in powers of δ, to obtain

$$\left.\begin{aligned} 0 &= R^{(0)}\frac{\partial\theta^{(0)}}{\partial x}+\nabla^4\psi^{(0)},\\ 0 &= \frac{\partial\psi^{(0)}}{\partial x}+\nabla^2\theta^{(0)},\\ \mathbf{U}^{(0)}.\nabla S^{(0)} &= \eta\nabla^2 S^{(0)}, \end{aligned}\right\} \tag{3.10}$$

and

$$\left.\begin{aligned} 0 &= R^{(1)}\frac{\partial\theta^{(0)}}{\partial x}+R^{(0)}\frac{\partial\theta^{(1)}}{\partial x}-P_S\eta\delta^{-1}\frac{\partial S^{(0)}}{\partial x}+\nabla^4\psi^{(1)},\\ 0 &= \frac{\partial\psi^{(1)}}{\partial x}+\nabla^2\theta^{(1)}. \end{aligned}\right\} \tag{3.11}$$

Higher-order corrections to the salt field will not be required. Thus the procedure is clear: (3.10a, b) must be solved to determine the basic eigenvalue $R^{(0)}$ and eigenfunctions $\theta^{(0)}$, $\psi^{(0)}$ (normalized as in (3.4)). Then $S^{(0)}$ can be calculated as a function of $\mathbf{U}_0$, η, and the correction $R^{(1)}$ to $R^{(0)}$ can be found by finding the solvability condition for the inhomogeneous system (3.11). If we multiply (3.11a) by $\psi^{(0)}$, (3.11b) by $\theta^{(0)}$ and integrate over a convective cell, we obtain

$$R^{(1)}\left\langle\psi^{(0)}\frac{\partial\theta^{(0)}}{\partial x}\right\rangle-R^{(0)}\left\langle\theta^{(1)}\frac{\partial\psi^{(0)}}{\partial x}\right\rangle+\langle\psi^{(1)}\nabla^4\psi^{(0)}\rangle+P_S\eta\delta^{-1}\left\langle S^{(0)}\frac{\partial\psi^{(0)}}{\partial x}\right\rangle = 0, \tag{3.12}$$

$$0 = \left\langle\theta^{(0)}\frac{\partial\psi^{(1)}}{\partial x}\right\rangle+\langle\theta^{(1)}\nabla^2\theta^{(0)}\rangle, \tag{3.13}$$

where

$$\langle\ldots\rangle = \frac{1}{k}\int_0^k\int_0^1\ldots dx\,dz.$$

However, from (3.10) we obtain the relations

$$R^{(0)}\left\langle\psi^{(1)}\frac{\partial\theta^{(0)}}{\partial x}\right\rangle+\langle\psi^{(1)}\nabla^4\psi^{(0)}\rangle = 0, \tag{3.14}$$

$$\left\langle\theta^{(1)}\frac{\partial\psi^{(0)}}{\partial x}\right\rangle+\langle\theta^{(1)}\nabla^2\theta^{(0)}\rangle = 0, \tag{3.15}$$

and further manipulation then yields

$$\delta R^{(1)} = P_S\eta\left\langle S^{(0)}\frac{\partial\psi^{(0)}}{\partial x}\right\rangle \tag{3.16}$$

if we write $S^{(0)} = S'-z+\frac{1}{2}$ so that $S' = 0$ at $z = 0$, 1 (3.10c) yields finally

$$\delta R^{(1)} = P_S\eta^2\langle|\nabla S'|^2\rangle. \tag{3.17}$$

Thus $\delta R^{(1)}$ is positive, and is proportional to the energy dissipation by the *perturbed* salt field S'. Equation (3.17) can be thought of as an expansion derived from the familiar power integral, which shows that in the steady-state buoyancy forces are balanced by viscous diffusion and dissipation in the salt field. Our sole remaining task, then, is to calculate S' from (3.10c) when η is small. Busse (1975) has provided numerical computations that could be adapted to determine this quantity for all

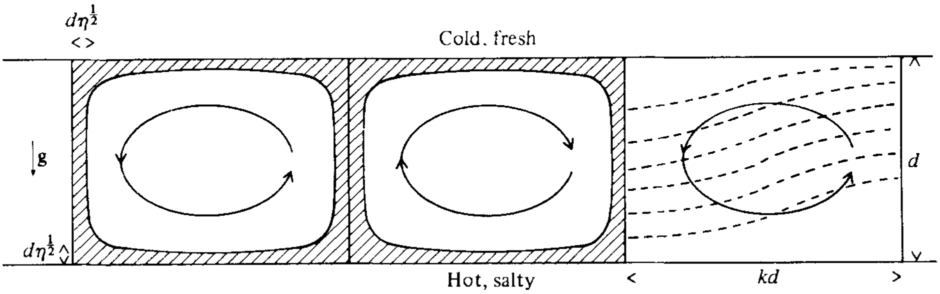

FIGURE 1. Sketch of the geometry and boundary-layer structure for the two-dimensional thermohaline problem at low ϵ, η. The left two cells show the region in which the salinity differs from its mean value, and the third shows the isotherms.

values of η: however, we can produce solutions in the rigid-boundary case using analytical techniques, and the same methods yield approximate answers in the free-boundary case that are valid to within a few per cent. The boundary-layer analysis also gives information on the magnitude of any terms neglected in the analysis.

When η is large in (3.10*c*), it is clear that far from any boundary, when length scales are $O(1)$, diffusion can be neglected. Thus in these regions $\mathbf{U}^{(0)}.\nabla S^{(0)} \simeq 0$, so that

$$S^{(0)} = S^{(0)}(\psi^{(0)}). \tag{3.18}$$

It can then be shown by integrating around each closed streamline of the flow that in fact $S^{(0)}$ is constant in the interior, and by symmetry $S^{(0)} = 0$ there. This interior solution does not satisfy the boundary conditions at $z = 0, 1$ and so there is a boundary layer round the edge of the cell whose nature depends on the boundary conditions (figure 1): we treat the free-boundary case first.

4. The free-boundary problem

It is easily shown, for example by comparison with the work of Roberts (1979), that the appropriate boundary-layer thickness is $O(\eta^{\frac{1}{2}})$ at both the horizontal and vertical boundaries. For the vertical layer at $x = 0$, for example, we define

$$\xi = x\eta^{-\frac{1}{2}}, \quad \xi = O(1), \quad \psi^{(0)} \sim \eta^{\frac{1}{2}}\xi\phi(z) \tag{4.1}$$

then the equation (3.10*c*) may be written

$$0 = -\xi\phi' \frac{\partial S^{(0)}}{\partial \xi} + \phi \frac{\partial S^{(0)}}{\partial z} + \frac{\partial^2 S^{(0)}}{\partial \xi^2}, \tag{4.2}$$

where the prime denotes differentiation with respect to z. This equation is to be solved with the boundary conditions that $\partial S^{(0)}/\partial \xi = 0$ at $\xi = 0$, $S^{(0)} \to 0$ as $\xi \to \infty$ and some condition at $z = 0$ that depends on the horizontal boundary layer there. Although this equation cannot be solved exactly, it can be used to deduce some results that elucidate the function of the boundary layer, and show how the global energy balance described in the last section is actually effected by local force balances.

We may readily deduce from (4.2), following Roberts (1979), that

$$\phi(z) \int_0^\infty S^{(0)} dz = \text{const.} = \tfrac{1}{2}\gamma, \quad \text{say}. \tag{4.3}$$

The term on the left-hand side is equal to $\eta^{\frac{1}{2}}$ times the flux of salt through the left-hand vertical plume as a fraction of the flux that is conducted when there is no motion. At leading order all the flux is advected through the vertical boundary layers, with diffusion playing a negligible role. Since there are two plumes, both transporting salt upwards, the total 'salt Nusselt number' is $\gamma\eta^{-\frac{1}{2}}$ and must equal the conductive salt flux at the boundary. Indeed, from (3.10*c*) we have

$$\int_0^k U_z^{(0)} S^{(0)} dx - \eta \int_0^k \frac{\partial S^{(0)}}{\partial z} dx = \text{const.} \tag{4.4}$$

and this constant must be $\gamma\eta^{\frac{1}{2}}$ from (4.3). Thus at $z = 0$,

$$\int_0^k \frac{\partial S^{(0)}}{\partial z} dx = -\gamma\eta^{-\frac{1}{2}}. \tag{4.5}$$

This flux can again be simply related to the quantity $\langle|\nabla S'|^2\rangle$ that determines $R^{(1)}$. Since $S^{(0)} = S' - z + \frac{1}{2}$ and (from (3.10*c*) again)

$$\langle S^{(0)} \nabla^2 S^{(0)} \rangle = 0 \tag{4.6}$$

we may readily show that

$$\langle|\nabla S'|^2\rangle = -\frac{1}{k}\int_0^k \frac{\partial S^{(0)}}{\partial z} dx\bigg|_{z=0} = \frac{\gamma}{k}\eta^{-\frac{1}{2}}. \tag{4.7}$$

Thus we see immediately that $\delta R^{(1)} = P_S \eta^{\frac{3}{2}} \gamma k^{-1}$ and the only remaining task is to calculate γ. It follows further from (4.7) that only the distribution of $S^{(0)}$ in the horizontal boundary layers need be considered, and this leads us to an approximate solution procedure.

Near $z = 0$, say, we may write

$$z = \eta^{\frac{1}{2}}\chi, \quad \chi = O(1), \quad \psi^{(0)} \sim \eta^{\frac{1}{2}}\chi f(x) \tag{4.8}$$

and then (3.10*c*) becomes, to leading order,

$$\chi f' \frac{\partial S^{(0)}}{\partial \chi} - f \frac{\partial S^{(0)}}{\partial x} = \frac{\partial^2 S^{(0)}}{\partial \chi^2} \tag{4.9}$$

with the boundary condition $S^{(0)} = \frac{1}{2}$ at $\chi = 0$, and (by symmetry) $\partial S/\partial x = 0$ at $x = k$, where the sign of $\psi^{(0)}$ is chosen so that fluid is flowing *towards* the plane $z = 0$ at $x = k$. Another boundary condition is needed as $\chi \to \infty$. Clearly, $S^{(0)} \to 0$ as $\chi \to \infty$ except in the neighbourhood of $x = k$, where a vertical salt plume impinges. If we ignore the effect of this plume, a similarity solution exists in the form

$$\left.\begin{aligned} S^{(0)} &= \tfrac{1}{2}\operatorname{erfc}(q), \\ q = \chi g(x); \quad g(x) &= \tfrac{1}{2}f(x)\bigg/\left(\int_x^k f(x')\,dx'\right)^{\frac{1}{2}}. \end{aligned}\right\} \tag{4.10}$$

where

Thus we have

$$\frac{\partial S^{(0)}}{\partial \chi}\bigg|_{\chi=0} = -g(x)\,\pi^{-\frac{1}{2}},$$

so that

$$\gamma = -\int_0^k \frac{\partial S^{(0)}}{\partial \chi}\bigg|_{\chi=0} dx = \pi^{-\frac{1}{2}}\int_0^k g(x)\,dx = \pi^{-\frac{1}{2}}\left(\int_0^k f(x)\,dx\right)^{\frac{1}{2}}. \tag{4.11}$$

Now for two-dimensional rolls between free boundaries, the eigenfunctions $\psi^{(0)}$, $\theta^{(0)}$

may be calculated explicitly (e.g. Chandrasekhar 1961): we find, using the normalization (3.4), that

$$\left.\begin{aligned} \psi^{(0)} &= c \sin \pi z \sin ax, \\ \theta^{(0)} &= \frac{ca}{a^2+\pi^2} \sin \pi z \cos ax, \\ R^{(0)} &= (a^2+\pi^2)^3/a^2, \\ a &= \pi/k, \quad c^2 = 4(a^2+\pi^2)/a^2. \end{aligned}\right\} \tag{4.12}$$

Thus $f(x) = c\pi \sin(\pi x/k)$ and so

$$\gamma = \left(\frac{2ck}{\pi}\right)^{\frac{1}{2}}. \tag{4.13}$$

The last relation gives a guide to the behaviour of γ as a function of k, but it is only an approximate solution which underestimates the true dissipation. A direct comparison with numerical results is possible in the case $k = 1$, since in that case the calculation carried out by Busse (1975) for convection in a magnetic field involves the determination of a function $(g_0(x,z)+x)$ which is determined by an equation that is identical (apart from a rotation through $\frac{1}{2}\pi$) to (3.10*c*). The quantity $E(A^*)-1$, where $A^* = c\eta^{-1}$ in the notation of the present paper, is equal to $\langle|\nabla S'|^2\rangle$, and for large A^* Busse finds

$$E(A^*)-1 \sim 1{\cdot}065A^{*\frac{1}{2}}, \quad A^* \to \infty. \tag{4.14}$$

Thus the exact result for $k = 1$ is

$$\gamma = 1{\cdot}065c^{\frac{1}{2}} \tag{4.15}$$

instead of the approximate value $\gamma = (2c/\pi)^{\frac{1}{2}}$ given by (4.13); the 20 % error is rather encouraging given the simplicity of the calculation, and presumably the accuracy would be even better for cells that have $k > 1$, since there would be less dissipation in the vertical plumes.

From (3.17) and (4.7) it is clear that $\delta = P_S\eta^{\frac{3}{2}}$ and we therefore choose P_S so that $P_S\eta^{\frac{3}{2}} = O(\epsilon^2)$ or

$$R_S\,\tau^{\frac{1}{2}} = O(\epsilon^{\frac{7}{2}}). \tag{4.16}$$

Note that we only require that $\tau \ll \epsilon$, and so, for any R_S, there is a sufficiently small τ for which (4.16) can be satisfied. From Busse, we have that the effect of finite Péclet number can be expressed (in our notation) in the form

$$\left.\begin{aligned} R_T &= R^{(0)}+\epsilon^2R_2+O(\epsilon^4), \\ R_2 &= \tfrac{1}{8}c^2(a^2+\pi^2)^2 = \tfrac{1}{2}(a^2+\pi^2)^3/a^2. \end{aligned}\right\} \tag{4.17}$$

Thus for $R_S\tau^{\frac{1}{2}} \sim \epsilon^{\frac{7}{2}}$ the expansion for R_T as a function of ϵ is

$$R_T = R^{(0)}+\epsilon^2R_2+\epsilon^{-\frac{3}{2}}R_S\,\tau^{\frac{1}{2}}\gamma k^{-1}, \tag{4.18}$$

where γ depends only on k. The consequences of this expression are investigated in §6. It is of course important to discover the leading-order corrections to (4.18) so as to determine the size of its domain of validity, and this entails solving the momentum equation in the boundary layer to show how the velocity field changes as a result of the horizontal salinity gradients. These questions are dealt with in the appendix. The conclusions obtained there are that the error in (4.18) is of the order of max $(\epsilon^4, \epsilon^2\eta^{\frac{1}{2}})$. In the next section we treat (in rather less detail) the case of rigid boundaries.

5. The expansion for rigid boundaries

In this case the thickness of the vertical plumes is still but the smaller velocities near $z = 0, 1$ mean that the layers there are of thickness $O(\eta^{\frac{1}{3}})$. For example, if near $z = 0$ we set

$$\chi = \eta^{-\frac{1}{3}}z, \quad \chi = O(1), \quad \psi^{(0)}(x,z) \sim \tfrac{1}{2}\eta^{\frac{2}{3}}\chi^2\zeta(x) \tag{5.1}$$

then the leading-order equation for $S^{(0)}$ in this region is

$$0 = -\frac{\chi^2}{2}\zeta'\frac{\partial S^{(0)}}{\partial \chi} + \chi\zeta\frac{\partial S^{(0)}}{\partial x} + \frac{\partial^2 S^{(0)}}{\partial \chi^2}. \tag{5.2}$$

It is then easy to see that the conductive salt flux at $z = 0$ is

$$-\int_0^k \frac{\partial S^{(0)}}{\partial z}\,dx = \beta\eta^{-\frac{1}{3}}; \quad \beta = O(1). \tag{5.3}$$

Now this flux is advected upwards in the vertical plumes at $x = 0$, k, as for the free boundary case. However, the plumes are of thickness $O(\eta^{\frac{1}{2}})$ and so in these regions $S^{(0)}$ has only to be of order $\eta^{\frac{1}{6}}$ in order to transport the flux (5.3). Thus the plumes can be neglected, both in their effect on the horizontal layers and in their contribution to the total dissipation. Hence the similarity arguments that were only approximate in the free-boundary case are here asymptotically exact.

Equation (5.2) has the following similarity solution satisfying all relevant boundary conditions:

$$\left.\begin{aligned} S^{(0)} &= \tfrac{1}{2}E(q), \quad E(q) = \frac{3^{\frac{2}{3}}}{\Gamma(\frac{1}{3})}\int_p^\infty \exp\left(\frac{-p^3}{3}\right)dp;\\ q &= \chi g(x), \quad g(x) = \zeta^{\frac{1}{2}}(x)\left(\frac{3}{2}\int_x^k \zeta^{\frac{1}{2}}(x')\,dx'\right)^{-\frac{1}{3}}. \end{aligned}\right\} \tag{5.4}$$

Then

$$\beta = -\int_0^k \frac{\partial S}{\partial \chi}\bigg|_{\chi=0} dx = \frac{1}{2}\cdot\frac{3^{\frac{2}{3}}}{\Gamma(\frac{1}{3})}\left\{\frac{3}{2}\int_0^k \zeta^{\frac{1}{2}}(x)\,dx\right\}^{\frac{2}{3}}. \tag{5.5}$$

Now $\zeta(x) = \lambda\sin ax$ for some $\lambda(a)$ (Chandrasekhar 1961) and so the integral can be evaluated to yield

$$\beta = \frac{3^{\frac{2}{3}}}{\Gamma(\frac{1}{3})}(\lambda k^2)^{\frac{1}{3}}(\tfrac{3}{4})^{\frac{2}{3}}\frac{\Gamma^{\frac{4}{3}}(\frac{3}{4})}{\Gamma^{\frac{2}{3}}(\frac{3}{2})} \tag{5.6}$$

for $a = \pi/k = 3{\cdot}117$ (corresponding to the minimum of $R^{(0)} = 1707{\cdot}76$), $\lambda = 10{\cdot}76\ldots$ (from numerical calculation) and $\beta = 0{\cdot}627$. When β is known, we have that $\delta R^{(1)} = P_S\eta^{\frac{5}{3}}\beta k^{-1}$ so at leading order the expansion of R_T is

$$R_T = R^{(0)} + \epsilon^2 R_2 + \epsilon^{-\frac{5}{3}}R_S\tau^{\frac{2}{3}}\beta k^{-1} + \ldots \tag{5.7}$$

and the conditions for the validity of this expression are that

$$R_S\tau^{\frac{2}{3}} \sim \epsilon^{\frac{11}{3}}. \tag{5.8}$$

The value of R_2 (which now depends on σ) is not known as a function of k in the rigid boundary case. It can, however, be shown to be positive.

6. Interpretation of results and discussion

The asymptotic results obtained in previous sections allow us to determine the minimum value ($R_{\min}$) of R_T for which steady convection is possible at given R_S. This value is correct to leading order if $R_S\tau^{\frac{1}{2}} \ll 1$ (free boundaries) or $R_S\tau^{\frac{2}{3}} \ll 1$ (rigid boundaries). In the free-boundary case we have (from (4.19))

$$R_{\min} = R^{(0)} + \frac{7}{4}\left[R_2^{-3}\left(\frac{3\gamma}{4k}\right)^4 R_S^4\tau^2\right]^{\frac{1}{7}}, \tag{6.1}$$

and the value of ϵ ($=\epsilon_{\min}$) for which this value of R_T occurs is given by

$$\epsilon_{\min}^2 = \tfrac{4}{7}\epsilon_0^2, \tag{6.2}$$

where ϵ_0 is the value of ϵ that would have been obtained at that value of R_T in the absence of salt ($R_S = 0$). Equation (6.2) shows plainly that the minimum in the (R_T, ϵ) curve occurs in a region in which the theory is valid. It is instructive to rearrange (6.1) to obtain the largest value of R_S compatible with steady convection for given $R_T - R_{(0)}$. We have

$$R_S \lesssim [\tfrac{4}{7}(R_T - R^{(0)})]^{\frac{7}{4}} R_2^{\frac{3}{4}} \left(\frac{3\gamma}{4k}\right)^{-1} \tau^{-\frac{1}{2}}, \tag{6.3}$$

and this shows clearly that, as $\tau \to 0$, R_S can be arbitrarily large and convection remain possible. In particular, convection can occur arbitrarily close to $R_T = R_0$ provided that $\kappa_S \to 0$ with all the other parameters kept fixed. This is an important qualitative difference from the rather similar problem (at least in the weakly nonlinear regime) of convection in the presence of a magnetic field treated by Busse (1975), where $R^{(0)}$ cannot be approached arbitrarily closely, whatever the differences between the diffusivities of temperature and magnetic field may be.

In the rigid boundary case the restrictions on R_S are even less severe. By analogy with (6.1), (6.2), (6.3) we have from (5.6)

$$R_{\min} = R^{(0)} + \frac{11}{6}\left[R_2^{-5}\left(\frac{5\beta}{6k}\right)^6 R_s^6\tau^4\right]^{\frac{1}{11}}, \tag{6.4}$$

$$\epsilon_{\min}^2 = \tfrac{6}{11}\epsilon_0^2, \tag{6.5}$$

$$R_S \lesssim [\tfrac{6}{11}(R_T - R^{(0)})]^{\frac{11}{6}} R_2^{\frac{5}{6}} \left(\frac{5\beta}{6k}\right)^{-1} \tau^{-\frac{2}{3}}. \tag{6.6}$$

All these calculations are irrelevant, of course, unless it can be shown that there is a range of values of R_S for which oscillatory convection is either impossible or unstable for $R_T < R_{\min}$. Unfortunately we have not been able to solve for the nonlinear oscillatory solutions analytically at finite amplitude, even when $\tau \ll 1$. We can, however, examine the linearized stability problem and the first nonlinear extension of it. We restrict our attention to free boundaries, for which the formulae involved take a relatively simple form. Linear theory tells us that, for infinitesimal oscillatory motions to be possible,

$$\begin{aligned} R_T &= \frac{\sigma}{\sigma+1}R_S + \frac{(\pi^2+a^2)^3}{a^2} \\ &= R^{(0)} + \frac{\sigma R_S}{\sigma+1}. \end{aligned} \tag{6.7}$$

and this is clearly much larger than $R_{\min}$ provided that σ is not too small. (We recall that for roll instabilities between free boundaries R_2 is independent of σ.) For hexagonal-type instabilities (not discussed here), $R_2 \sim \sigma^{-2}$ for $\sigma \to 0$: the analogue of (6.1) in this case is not known, but we expect $R_{\min}$ to decrease as σ decreases and the outcome of the competition is not obvious. In spite of this, it is clear that finite-amplitude steady motion can be found at values of R_T well below the value for which infinitesimal oscillations are possible.

Can finite-amplitude oscillations occur at a lower R_T than that given by (6.7)? We can provide a partial answer by examining the quantity δR_T defined by the equation

$$R_T = R^{(0)} + \frac{\sigma R_S}{\sigma+1} + A^2 \delta R_T, \tag{6.8}$$

where A is a measure of the amplitude of the oscillation. If $\delta R_T > 0$, then we have supercritical instability and we would then be surprised if the (R_T, A) curve doubled back on itself. If $\delta R_T < 0$, on the other hand, the instability is subcritical and (6.7) is irrelevant. Presumably there is some minimum value of R_T at which oscillations occur, but its relation to $R_{\min}$ is not known.

It is well known that, for small R_S, $\delta R_T > 0$. The value of R_S for which δR_T changes from positive to negative has been given in concise form by Da Costa *et al.* (1981). From their equations (15), (17), (18), it is easily seen that the critical value $R_S^{(c)}$ of R_S is, to leading order,

$$R_S^{(c)} = \left[\frac{2(1+\sigma)^2}{\sigma}\right] \frac{R^{(0)}}{\pi^2 + a^2} \tag{6.9}$$

for $\tau \ll 1$ and $\sigma \gg \tau$. (The latter condition is necessary in any event for the validity of the asymptotic analysis.) The function of σ appearing in the brackets in (6.9) is at least 8 and so for any value of σ,

$$R_S < \frac{8R^{(0)}}{\pi^2 + a^2} \tag{6.10}$$

is sufficient for supercritical bifurcation of the oscillatory mode. Of course this value of R_S is within the scope of the theory when $\tau \to 0$. Thus we expect a wide range of R_S, restricted by (6.10), in which as R_T is increased the first possible instability of the conduction solution is to a finite-amplitude steady mode. Presumably the fixed-boundary problem will yield similar results, although the critical values of R_T are not known in closed form in this case.

7. A tentative approach for larger R_S

The analysis outlined above only applies when (in the free boundary case) $R_S \tau^{\frac{1}{2}} \ll 1$, and the minimum value of R_T occurs for small ϵ. Huppert & Moore have observed the same general behaviour of the R_T curve for larger values of R_S; however, in their results the turn-round occurs when ϵ is $O(1)$ or greater, and when not only the salt, but also the thermal field, are significantly changed by the fluid motion. They observe a straight line law of the form

$$R_{\min} = A + BR_S \tag{7.1}$$

for constants A and B when R_S is varied and the other parameters kept fixed. This differs from the relationship (6.1), and indeed no close agreement is to be expected since $\tau = 0{\cdot}1$ for the models studied by HM and $\tau^{\frac{1}{2}}$ is not very small. However, a careful examination of Huppert & Moore's results (their figure 14) suggests that $R_{\min}$ falls below the line (7.1) for small R_S: so a match to a relation of the form (6.1) (for which $R_{\min}$ is a convex function of R_S) is not ruled out.

We can construct a rough model to show why $R_{\min}$ might depend linearly upon R_S when the latter is large. We begin with the 'power integral', valid for steady solutions of (2.1)–(2.3), namely

$$R_T\langle|\nabla\theta|^2\rangle - \tau R_S\langle|\nabla S|^2\rangle = \langle|\nabla \mathbf{U}|^2\rangle. \tag{7.2}$$

It has been noted often that (at least in the two-dimensional geometry) the form of $\mathbf{U}$ does not depend much on its amplitude in the steady state. Letting the amplitude be represented by the parameter V, with some appropriate scalings, we can represent the viscous dissipation by the approximate relation

$$\langle|\nabla \mathbf{U}|^2\rangle = V^2. \tag{7.3}$$

If we suppose that $\tau \ll 1$, the salt Péclet number $|\mathbf{U}|d/\kappa_S = O(V\tau^{-1})$ is large in the regime of interest. We then know that $\langle|\nabla S|^2\rangle$ depends only on the thickness of the boundary layer, which is proportional to $(V/\tau)^{-\frac{1}{2}}$. Thus we set

$$\langle|\nabla S|^2\rangle = A_1\tau^{-\frac{1}{2}}V^{\frac{1}{2}}, \tag{7.4}$$

where A_1 is of order unity. To obtain the *ansatz* for $\langle|\nabla\theta|^2\rangle$ we must model a greater range of behaviour. For large V,

$$\langle|\nabla\theta|^2\rangle = A_1V^{\frac{1}{2}}, \tag{7.5}$$

and for small V

$$\langle|\nabla\theta|^2\rangle = B_1V^2 - C_1V^4 + \dots. \tag{7.6}$$

A simple function of V satisfying both these requirements is

$$\langle|\nabla\theta|^2\rangle = E[(1+FV^2)^{\frac{1}{4}} - (1+GV^2)^{\frac{1}{4}}], \tag{7.7}$$

where E,F,G are defined by

$$E(F^{\frac{1}{4}} - G^{\frac{1}{4}}) = A_1, \quad E(F-G) = 4B_1, \quad E(F^2-G^2) = \frac{32C_1}{3}. \tag{7.8}$$

(Equations (7.8) cannot be satisfied for arbitrary A_1, B_1, C_1: however, more complicated functions can be constructed in these cases. The results presented below do not depend on the function that is chosen.) Our approximate expression for R_T as a function of R_S and V is then

$$R_T = \frac{V^2 + R_S\tau^{\frac{1}{2}}V^{\frac{1}{2}}A_1}{E[(1+FV^2)^{\frac{1}{4}} - (1+GV^2)^{\frac{1}{4}}]}. \tag{7.9}$$

It can easily be shown that this function has all the correct power law dependencies in cases where the latter are known. For example, the mean field result $(R_T - \tau^{\frac{1}{2}}R_S) \sim V^{\frac{3}{2}}$ holds when each side is large, and the relation (4.19) is qualitatively reproduced with V replacing ϵ. If we now seek the minimum R^* of the function on the right-hand side of (7.9), we may expect its behaviour to resemble that of $R_{\min}$ as a function of $R_S\tau^{\frac{1}{2}}$. The minimum cannot be found in closed form for all $R_S\tau^{\frac{1}{2}}$: for small $R_S\tau^{\frac{1}{2}}$ we have, from earlier sections,

$$R^* = P + Q[R_S\tau^{\frac{1}{2}}]^{\frac{4}{7}}, \tag{7.10}$$

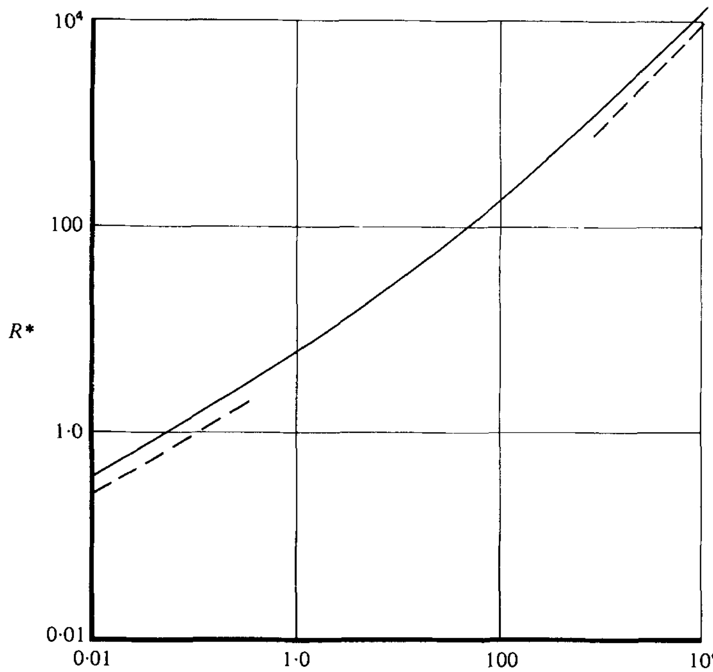

FIGURE 2. Graph of $R^* - R^{(0)}$ found from the expression (7.9) as a function of $R_S\tau^{\frac{1}{2}}$ (log–log plot). The transition from the 4/7 power law to the linear power law is shown clearly. The dotted lines are straight segments with the appropriate respective gradients.

which models (6.1), where P and Q are constants. For large $R_S\tau^{\frac{1}{2}}$ we find that

$$R^* = SR_S\tau^{\frac{1}{2}} + \text{smaller terms}, \quad S \text{ constant} \tag{7.11}$$

and this occurs for $V \sim (R_S\tau^{\frac{1}{2}})^{\frac{1}{2}}$, or, alternatively, for a thermal Nusselt number $\sim (R_S\tau^{\frac{1}{2}})^{\frac{1}{2}}$. Figure 2 shows R^* as a function of $R_S\tau^{\frac{1}{2}}$ (log–log plot) in the simple case $A_1 = E = F = 1$, $G = 0$, which is not unrepresentative. The transition between the two power laws given by (7.10) and (7.11) is shown clearly. We could attempt to quantify (7.11) by trying to match the coefficients with the results of HM, but feel that it would be overstretching a theory that only pretends to qualitative predictions.

We have thus shown that the crucial requirement for subcritical steady convection to exist at values of R_T below that for which oscillatory convection is possible is that κ_S be small (compared to κ_T). Indeed, convection can take place at values of R_T arbitrarily close to that needed for convection when $R_S = 0$ provided that $\kappa_S \to 0$ with R_S remaining fixed. (This result goes some way towards explaining the energy stability result of Shir & Joseph (1968), who show that if $R_S > 0$ it does not appear in any criterion for stability based on the energy method.) The analysis resembles that of Busse (1975) but we elucidate the structure of the boundary layers somewhat, following Roberts (1979), and show how the analysis can be extended into other parameter ranges. A qualitative link can be established, via an approximate theory, with the results of Huppert & Moore for large $R_S\tau^{\frac{1}{2}}$, and the problem with fixed top and bottom boundaries can be shown to have similar properties.

The boundary-layer analysis makes clear the important distinction between subcritical thermohaline convection and the related problem of convection in a magnetic field (Busse 1975; Proctor & Galloway 1979). In the latter case, the magnetic flux is pushed into a rope or sheet that is similar, when its dynamic effect is weak, to the salt plume discussed in this paper. However, because of the nonlinear character of the magnetic (Lorentz) forces, the flux sheet becomes more dynamically active as it gets thinner, in contrast to the salt plumes in the present study. The problem of convection in a rotating fluid layer can also be studied (when the Prandtl number is small) by similar asymptotic methods and turns out to be much like the magnetic problem in its finite-amplitude behaviour.

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Appendix. The effect on the velocity field of horizontal salt gradients

The analysis given in the main body of the paper is quite sufficient to obtain the leading-order behaviour. If a further expansion is desired, however, it is necessary to determine explicitly the change $\delta \mathbf{U}^{(1)}$ in the velocity field caused by the salt field. Clearly the salt field only affects the motion when its horizontal gradient is large, and so we need only consider the vertical plumes. Near $x = 0$ for example the vorticity field $\omega^{(1)}$ can be divided into two parts: we write

$$\omega^{(1)} = \omega_I + \tilde{\omega}(\xi, z), \quad \xi = \eta^{-\frac{1}{2}}x, \tag{A 1}$$

where $\tilde{\omega}$ is non-zero only in the boundary layer, and ω_I is the 'internal' (non-boundary-layer) part; clearly $\omega_I + \tilde{\omega} = 0$ at $x = \xi = 0$. Then equation (3.5) can be written, at leading order in the boundary layer,

$$0 = P_S \eta^{\frac{1}{2}} \frac{\partial S^{(0)}}{\partial \xi} + \eta^{-1}\delta \frac{\partial^2 \tilde{\omega}}{\partial \xi^2}, \tag{A 2}$$

where $\tilde{\omega} \to 0$ as $\xi \to \infty$. Integrating this equation then shows that

$$\omega_I(x=0) = -\tilde{\omega}(\xi = 0) = -P_S \eta^{\frac{3}{2}} \delta^{-1} \int_0^\infty S^{(0)}\, d\xi. \tag{A 3}$$

In (A 2) we have neglected the boundary-layer part of the term $-R^{(0)}\partial\theta^{(1)}/\partial x$ since it can be shown to be of order η^2 compared to the viscous term, and therefore small. The vorticity field gives rise to a boundary-layer velocity field $\tilde{\mathbf{U}}$ which is of order $\eta^{\frac{1}{2}}$ compared to the 'interior' velocity field $\mathbf{U}_I$ and can therefore be ignored.

Thus the leading-order correction to the velocity field ($\delta \mathbf{U}^{(1)}$) can be found as the solution to the following problem:

$$\left.\begin{aligned} 0 &= R^{(0)}\frac{\partial \theta_I}{\partial x} + \nabla^4 \psi_I + R^{(1)}\frac{\partial \theta^{(0)}}{\partial x}, \\ 0 &= \frac{\partial \psi_I}{\partial x} + \nabla^2 \theta_I \end{aligned}\right\} \tag{A 4}$$

(no term in $S^{(0)}$ appears in the interior as its horizontal gradient is zero there). These equations are to be solved subject to the standard boundary,conditions except that, at $x = 0$, the condition $\partial^2\psi_I/\partial x^2 = 0$ is replaced (from (A 3)) by

$$\partial^2\psi_I/\partial x^2\Big|_{x=0} = +P_S\eta^{\frac{3}{2}}\delta^{-1}\int_0^\infty S^{(0)}\,d\xi, \tag{A 5}$$

and similarly at $x = k$. Now we know from (4.3) that

$$\frac{\partial\psi^{(0)}}{\partial x}\Big|_{x=0}\int_0^\infty S^{(0)}dx = \gamma/2, \tag{A 6}$$

and so the condition (A 5) becomes

$$\partial^2\psi/\partial x^2 = +\frac{P_S\eta^{\frac{3}{2}}\gamma}{2\psi_x^{(0)}\delta}, \quad x = 0,\, k. \tag{A 7}$$

This condition is very similar in form to ones obtained by Roberts (1979) in the Bénard convection problem. $R^{(1)}$ can then be determined from a solvability condition in the usual way, and its value agrees with that derived earlier by an independent method. The techniques used in this appendix could be used even if $P_S\eta^{\frac{3}{2}}$ is of order unity (though no analytic solution is known in that case) since the velocity field in the plumes is always of 'interior' type at leading order.

We can see, then, that the errors in the expansion we have described in the body of the paper are either due to finite Péclet number (ϵ^4), finite $\delta(P_S^{\frac{3}{2}}\eta^3)$, interaction of finite Péclet number and finite ($\epsilon^2P_S\eta^{\frac{3}{2}}$) and boundary-layer corrections to the salt field (at most $\epsilon^2\eta^{\frac{1}{2}}$). All these are small compared with ϵ^2 and can therefore be neglected provided that ϵ^2 and $P_S\eta^{\frac{3}{2}}$ are small. Similar considerations clearly hold for the rigid boundary case.

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